A possible N soliton solution for a nonlinear optics equation

1972 J. Phys. A: Gen. Phys. 5 L122

(http://iopscience.iop.org/0022-3689/5/11/015)

J. Phys. A: Gen. Phys., Vol. 5, November 1972. Printed in Great Britain. © 1972.

LETTER TO THE EDITOR

A possible N soliton solution for a nonlinear optics equation

J D GIBBON and J C EILBECK

Department of Mathematics, UMIST, Sackville Street, Manchester M60 1QD, UK

MS received 15 August 1972

Abstract. We suggest a possible N soliton solution for an equation of nonlinear optics (and hence, by a transformation, for the sine-Gordon equation). The proposed solution, which has been tested for $N \leqslant 3$, is a combination of simple functions similar to the N soliton solution of the Korteweg–de Vries equation.

Recently Hirota (1971) has published a simple, but exact, N soliton solution of the Korteweg–de Vries (KV) equation

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

It is tempting to assume that other nonlinear partial differential equations related in some way to the KV equation have similar simple N soliton solutions. In this letter we propose an N soliton solution to one such equation of importance in nonlinear optics and other branches of physics. Our solution has been tested for $N \leqslant 3$ but we have not yet obtained a general proof for all N. The equation we consider is most simply given as a coupled set of partial differential equations:

$$E_x + E_t = \alpha P \tag{2a}$$

$$P_t = EN \tag{2b}$$

$$N_t = -EP. \tag{2c}$$

These are a dimensionless form of a set of equations describing the evolution of the envelope of a resonant carrier wave interacting with a medium of two-level atoms (Lamb 1971, to be referred to as I). E and P are the electric field and microscopic polarization respectively, N is a measure of the atomic inversion, and α is a dimensionless constant proportional to the atomic density. The constant of integration $P^2 + N^2$ is unity and the boundary conditions are $E, P \to 0$, $N \to (-1)$ as $x \to \pm\infty$. As these equations are important in the theory of selfinduced transparency (McCall and Hahn 1969) we shall refer to equations (2) as the SIT equations. The similarity between (1) and (2) is that for steady state solutions of (2), E^2 satisfies the steady state KV equation (1).

We can write the SIT equations in several forms. On elimination of P and N we have

$$(E_{tt} + E_{xt})^2 = \alpha^2 E^2 - E^2(E_t + E_x)^2. \tag{3}$$

Alternatively we can make the substitution

$$E = \frac{\partial \sigma}{\partial t}, \qquad P = -\sin \sigma, \qquad w = \alpha^{1/2} t, \qquad v = \alpha^{1/2}(t - 2x)$$

to get (2) in the form

$$\sigma_{vv} - \sigma_{ww} = \sin \sigma. \tag{4}$$

This is the sine-Gordon equation (Rubinstein 1970) which appears in many branches of physics (see the references cited in I). Given a one-soliton solution to (4) it is, in principle, possible to construct higher soliton solutions by means of repeated Bäkelund transformations (I) but in practice the labour involved is prohibitive for $N > 3$.

We propose the following N soliton solution for the SIT equations:

$$E^2 = 4\frac{\partial^2}{\partial t^2}\ln f \tag{5}$$

$$f = \det |M| \tag{6}$$

where the i, jth element of the $N \times N$ matrix M has the form

$$M_{ij} = \frac{2(E_i E_j)^{1/2}}{E_i + E_j}\{\exp(\theta_i) + (-1)^{i+j}\exp(-\theta_j)\} \tag{7}$$

and

$$\theta_i = \omega_i t - \kappa_i x + \delta_i$$

$$\omega_i = \tfrac{1}{2}E_i, \qquad \kappa_i/\omega_i = 1 + 4\alpha E_i^{-2}. \tag{8}$$

E_i and δ_i are arbitrary constants determining the amplitude and phase, respectively, of the ith soliton. The E_i are assumed to be all different, but not necessarily positive, such that $|E_i| \neq |E_j|$ for $i \neq j$. The sign of E in (5) is defined to be the same as f.

We have not been able to prove that the solution given above is an exact solution to the SIT equations for all N, but as described below we have checked that the solution is correct for $N \leqslant 3$. In view of the simplicity of the solution it is plausible that is it exact for all finite N. A general proof would be more difficult than that needed for the KV solution (Hirota 1971) since the matrix M_{ij} (equation (7)) has no leading diagonal and two exponential terms.

Further physical insight into the nature of our solution can be gained by examining the $N = 1, 2, 3$ forms in detail. For $N = 1$ we have from equations (5)–(8)

$$E(x, t) = E_1 \operatorname{sech} \theta_1. \tag{9}$$

This is the well known 2π pulse of selfinduced transparency (McCall and Hahn 1969). For $N = 2$ our solution is

$$E = \left(\frac{E_1^2 - E_2^2}{E_1^2 + E_2^2}\right)\frac{E_1 \operatorname{sech}\theta_1 + E_2 \operatorname{sech}\theta_2}{1 - B_{12}(\tanh\theta_1 \tanh\theta_2 - \operatorname{sech}\theta_1 \operatorname{sech}\theta_2)} \tag{10}$$

where $B_{12} = 2E_1E_2/(E_1^2 + E_2^2)$. This is the Lamb two-soliton solution obtained by Bäkelund transformations (I). (The solution (10) is a 0π or 4π pulse depending on the relative signs of E_1 and E_2.) For $E_1 > E_2$ and $t \to \pm\infty$ we have from (10)

$$E \to E_1 \operatorname{sech}(\theta_1 \pm \beta_{12}) + E_2 \operatorname{sech}(\theta_2 \mp \beta_{12}) \tag{11}$$

with

$$\beta_{12} = \tanh^{-1} B_{12} = \ln\left|\frac{E_1 + E_2}{E_1 - E_2}\right|.$$

It is interesting to compare (11) with Hirota's two soliton solution for $t \to \pm\infty$

$$E^{\mathrm{K}} \to E_1{}^{\mathrm{K}} \operatorname{sech}(\theta_1 \genfrac{}{}{0pt}{}{+\,0}{-\beta_{12}}) + E_2{}^{\mathrm{K}} \operatorname{sech}(\theta_2 \genfrac{}{}{0pt}{}{-\beta_{12}}{+\,0})$$

where $(E^{\mathrm{K}})^2 = u$ and $\theta_i = E_i{}^{\mathrm{K}}x - (E_i{}^{\mathrm{K}})^3 t$ in equation (1). Note that the overall phase shift in (11) is exactly twice that occuring in the KV solution.

For $N = 3$ our full solution is rather lengthy but the asymtotic form is simple: for $t \to \pm\infty$ and $E_1 > E_2 > E_3$ we have

$$E \to E_1 \operatorname{sech}(\theta_1 \pm \beta_{12} \pm \beta_{13}) + E_2 \operatorname{sech}(\theta_2 \mp \beta_{12} \pm \beta_{23}) + E_3 \operatorname{sech}(\theta_3 \mp \beta_{13} \mp \beta_{23}) \tag{12}$$

with

$$\beta_{ij} = \ln \left| \frac{E_i + E_j}{E_i - E_j} \right| .$$

An interesting feature of (12) is that the phase shifts resulting from a three-soliton collision are a *linear* sum obtained from taking an appropriate combination of two-soliton collisions (equation (11)). This is obviously correct if the three solitons are spaced such that only two overlap at any time, but rather surprising when the three solitons collide simultaneously. To test the accuracy of this solution we have compared our analytic solution for three solitons to the numerical solution of equations (2) with initial conditions of three solitons widely spaced with amplitudes and phases fixed to give a simultaneous collision. Our analytic solution fitted the numerical solution for the whole collision process to within the accuracy of the numerical integration procedure ($\leqslant 0{\cdot}5\%$). We take this as evidence that our three-soliton solution is probably an exact solution of equations (2).

The form of (12) suggests that the phase changes of an N soliton collision can be simply calculated by the assumption that the multiple collision is equivalent to a series of two-soliton collisions in any order. (A similar statement appears to hold for the KV equation.)

Our solution can also be used to predict pulse break-up. If, for instance, we start with a 6π pulse, the amplitude of the three solitons into which this pulse separates can be calculated by the use of conservation equations (I). This leaves three free parameters, the δ_i in equation (8), which can be used to fit the shape of the initial pulse. The phases of the three solitons, after separation, will be the δ_i plus an appropriate sum of the β_{ij}.

We would like to thank Dr R K Bullough of UMIST and Dr D Griffel of the University of Bristol for stimulating conversations relating to the nonlinear optics equations and the KV equation respectively. One of us (J C Eilbeck) wishes to thank the SRC for the financial support of a research assistantship.

References

Hirota R 1971 *Phys. Rev. Lett.* **27** 1192–4
Lamb G L Jr 1971 *Rev. mod. Phys.* **43** 99–124
McCall S L and Hahn E L 1969 *Phys. Rev.* **183** 457–85
Rubinstein J 1970 *J. math. Phys.* **11** 258–66